

# On the simplified path integral on spheres

Fiorenzo Bastianelli<sup>1,3,4,a</sup> , Olindo Corradini<sup>2,3,4,b</sup> 

<sup>1</sup> Dipartimento di Fisica ed Astronomia, Università di Bologna, Via Irnerio 46, 40126 Bologna, Italy

<sup>2</sup> Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università degli Studi di Modena e Reggio Emilia, Via Campi 213/A, 41125 Modena, Italy

<sup>3</sup> INFN, Sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy

<sup>4</sup> Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, 14476 Golm, Germany

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**Abstract** We have recently studied a simplified version of the path integral for a particle on a sphere, and more generally on maximally symmetric spaces, and proved that Riemann normal coordinates allow the use of a quadratic kinetic term in the particle action. The emerging linear sigma model contains a scalar effective potential that reproduces the effects of the curvature. We present here further details of the construction, and extend its perturbative evaluation to orders high enough to read off the type-A trace anomalies of a conformal scalar in dimensions  $d = 14$  and  $d = 16$ .

## 1 Introduction

Path integrals for point particles find useful applications in worldline treatments of quantum field theories. In particular, path integrals for particles on curved spaces allow one to study gravitationally interacting field theories. In this paper, after reviewing the simplified path integral for a nonrelativistic particle on a sphere that has been introduced recently in [1], by presenting further details of its construction, we extend its perturbative calculation to orders high enough to be able to read off the type-A trace anomalies of a conformal scalar field in dimensions  $d = 14$  and  $d = 16$ .

The standard action of a nonrelativistic particle has the form of a nonlinear sigma model in one dimension. The nonlinearities present in the kinetic term make the definition of the path integral rather subtle, carrying the necessity of specifying a regularization scheme together with the fixing of corresponding finite counterterms. The latter are needed for specifying a well-defined quantum theory; see [2–5] for the known regularization schemes. The development of those regularization schemes was prompted by the

desire of extending the quantum mechanical method of computing chiral anomalies [6–8] to trace anomalies [9, 10]. A comprehensive account may be found in the book [11].

A simplified version of the path integral for the case of maximally symmetric spaces, like spheres, has been discussed and proved recently in [1]. It builds on an old proposal [12] of constructing the path integral by making use of Riemann normal coordinates. These special coordinates are supposed to make consistent the replacement of the nonlinear sigma model by a linear one. At the same time the inclusion of a suitable effective scalar potential is shown to reproduce the effects of the curvature. That this is indeed possible was proved in [1] for the case of maximally symmetric spaces, leaving the more difficult question of its validity on arbitrary geometries unsettled. In the present paper we review the construction on maximally symmetric spaces, and present a detailed perturbative evaluation of the path integral, which in particular allows us to identify the trace anomalies of a conformal scalar field in dimensions  $d = 14$  and  $d = 16$ . The maximally symmetric background gives information on the so-called type-A trace anomaly [13], which is proportional to the Euler density of the curved background. Other methods for identifying the type-A trace anomalies in higher dimensions are probably more efficient, see for example [14–17], but the path integral construction is certainly more flexible, allowing in principle for the calculation of other observables, as exemplified by the various applications of the worldline formalism (see [18] for a review in flat space, and [19–28] for extensions to curved spaces). In any case, we also apply these alternative methods to check our final anomaly coefficients.

We start our paper with Sect. 2 where, by using Riemann normal coordinates on maximally symmetric spaces, we prove that the Schrödinger equation (the heat equation in our euclidean convention) for the transition amplitude can be simplified, so to have a corresponding simplified version of the path integral that generates its solutions. In Sect. 3

<sup>a</sup> e-mail: [fiorenzo.bastianelli@bo.infn.it](mailto:fiorenzo.bastianelli@bo.infn.it)

<sup>b</sup> e-mail: [olindo.corradini@unimore.it](mailto:olindo.corradini@unimore.it)

we set up the perturbative expansion of the simplified path integral, and proceed to evaluate the transition amplitude at coinciding points, as needed for identifying one-loop effective actions in scalar QFT through worldlines. In particular, we calculate all the terms that are needed to identify the trace anomalies for space-time dimensions  $d \leq 16$ . These anomalies are extracted in Sect. 4, and recomputed in Sect. 5 with the alternative methods mentioned earlier to show the consistencies of these different approaches. Eventually, we present our conclusions and outlook in Sect. 6.

## 2 Transition amplitude and path integral on spheres

The classical dynamics of a nonrelativistic particle of unit mass in a curved  $d$ -dimensional space is described by the lagrangian

$$L(x, \dot{x}) = \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j, \quad (2.1)$$

where  $g_{ij}(x)$  is the metric in an arbitrary coordinate system and  $\dot{x}^i = \frac{dx^i}{dt}$ . The corresponding hamiltonian reads

$$H(x, p) = \frac{1}{2} g^{ij}(x) p_i p_j, \quad (2.2)$$

where the  $p_i$  are the momenta conjugate to  $x^i$ . Upon canonical quantization the classical hamiltonian identifies a quantum hamiltonian operator

$$\hat{H}_\xi(\hat{x}, \hat{p}) = \frac{1}{2} g^{-\frac{1}{4}}(\hat{x}) \hat{p}_i g^{\frac{1}{2}}(\hat{x}) g^{ij}(\hat{x}) \hat{p}_j g^{-\frac{1}{4}}(\hat{x}) + \frac{\xi}{2} R(\hat{x}), \quad (2.3)$$

where the ordering ambiguities between the  $\hat{x}^i$  and  $\hat{p}_i$  operators have been partially fixed by requiring background general coordinate invariance (here  $g(x) \equiv \det g_{ij}(x)$ ). The remaining ambiguities are parametrized by the free coupling constant  $\xi$ , which multiplies the scalar curvature  $R$ . Interesting values of this coupling are  $\xi = 0$ , which defines the minimal coupling,  $\xi = \frac{d-2}{4(d-1)}$  for the conformally invariant coupling in  $d$  dimensions, and  $\xi = \frac{1}{4}$ , which allows for a supersymmetrization of the model (it appears in the square of the Dirac operator). For simplicity, we will set  $\xi = 0$  in the following discussion, inserting the nonminimal coupling through a scalar potential, when needed.

We are interested in studying the evolution operator in euclidean time  $\beta$  (the heat kernel)

$$\hat{K}(\beta) = e^{-\beta \hat{H}_0}, \quad (2.4)$$

which satisfies the equation

$$-\frac{\partial \hat{K}(\beta)}{\partial \beta} = \hat{H}_0 \hat{K}(\beta), \quad (2.5)$$

$$\hat{K}(0) = \mathbb{1}. \quad (2.6)$$

It is convenient to use position eigenstates

$$\hat{x}^i |x\rangle = x^i |x\rangle, \quad (2.7)$$

normalized as

$$\langle x | x' \rangle = \frac{\delta^{(d)}(x - x')}{\sqrt{g(x)}}, \quad (2.8)$$

so that the resolution of the identity is written as

$$\mathbb{1} = \int d^d x \sqrt{g(x)} |x\rangle \langle x|. \quad (2.9)$$

Using them, one recognizes that the wave functions  $\psi(x) = \langle x | \psi \rangle$ , corresponding to vectors  $|\psi\rangle$  of the Hilbert space, are scalars under arbitrary change of coordinates. In particular the matrix element of the evolution operator between these position eigenstates gives a transition amplitude

$$K(x, x'; \beta) = \langle x | e^{-\beta \hat{H}_0} | x' \rangle, \quad (2.10)$$

which behaves as a biscalar under arbitrary change of coordinates, i.e. a scalar at both points  $x$  and  $x'$ . It satisfies the heat equation (we use units with  $\hbar = 1$ )

$$-\frac{\partial}{\partial \beta} K(x, x'; \beta) = -\frac{1}{2} \nabla_x^2 K(x, x'; \beta), \quad (2.11)$$

$$K(x, x'; 0) = \frac{\delta^{(d)}(x - x')}{\sqrt{g(x)}}, \quad (2.12)$$

where  $\nabla_x^2$  is the scalar laplacian  $\nabla^2 = \frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j$  acting on the  $x$  coordinates. This equation corresponds precisely to the matrix elements of the operatorial equation (2.5) between position eigenstates. Its solution may be given a well-defined path integral representation in terms a nonlinear sigma model action [11].

In order to simplify the heat equation and the corresponding path integral, we choose to work with Riemann normal coordinates, which are reviewed in Appendix A. We first transform the transition amplitude into a bidensity by defining

$$\bar{K}(x, x', \beta) = g^{\frac{1}{4}}(x) K(x, x', \beta) g^{\frac{1}{4}}(x'), \quad (2.13)$$

so that Eq. (2.11) takes the form

$$-\frac{\partial}{\partial \beta} \bar{K}(x, x'; \beta) = -\frac{1}{2} g^{\frac{1}{4}}(x) \nabla_x^2 \left( g^{-\frac{1}{4}}(x) \bar{K}(x, x'; \beta) \right), \quad (2.14)$$

$$\bar{K}(x, x'; 0) = \delta^{(d)}(x - x'). \quad (2.15)$$

One may evaluate the differential operator appearing on the right hand side of Eq. (2.14) to obtain the identity

$$-\frac{1}{2}g^{\frac{1}{4}}\nabla^2g^{-\frac{1}{4}} = -\frac{1}{2}\partial_i g^{ij}\partial_j + V_{\text{eff}}, \quad (2.16)$$

where in the first term derivatives act through, while the effective scalar potential is given by

$$V_{\text{eff}} = -\frac{1}{2}g^{-\frac{1}{4}}\partial_i\sqrt{g}g^{ij}\partial_jg^{-\frac{1}{4}}, \quad (2.17)$$

where all derivatives stop after acting on the last function. The heat equation (2.14) now reads more explicitly as

$$-\frac{\partial}{\partial\beta}\bar{K}(x, x'; \beta) = \left(-\frac{1}{2}\partial_i g^{ij}(x)\partial_j + V_{\text{eff}}(x)\right)\bar{K}(x, x'; \beta). \quad (2.18)$$

At this stage we are ready to use the properties of Riemann normal coordinates, centered at the point  $x'$ , to show that this heat equation simplifies further to

$$-\frac{\partial}{\partial\beta}\bar{K}(x, x'; \beta) = \left(-\frac{1}{2}\delta^{ij}\partial_i\partial_j + V_{\text{eff}}(x)\right)\bar{K}(x, x'; \beta), \quad (2.19)$$

as on maximally symmetric space, in Riemann normal coordinates, one may replace the metric  $g^{ij}(x)$  appearing in the term  $\partial_i g^{ij}(x)\partial_j$  by the constant metric  $\delta^{ij}$ . Note that the heat kernel equation (2.19) contains now an hamiltonian operator

$$H = -\frac{1}{2}\delta^{ij}\partial_i\partial_j + V_{\text{eff}}(x), \quad (2.20)$$

which is interpreted as that of a particle on a flat space (in cartesian coordinates) interacting with an effective scalar potential  $V_{\text{eff}}$  of quantum origin (it would be proportional to  $\hbar^2$  in arbitrary units).

For the replacement of  $g^{ij}(x)$  with  $\delta^{ij}$  to be valid, one must show that

$$\left(\partial_i g^{ij}(x)\partial_j - \delta^{ij}\partial_i\partial_j\right)\bar{K}(x, x'; \beta) = 0. \quad (2.21)$$

To see this, we recall that  $x' = 0$  is the chosen origin of the Riemann normal coordinates, and using the inverse metric given in (A.9) we find that the equation that we must verify takes the form

$$\left(h(x)P^{ij}(x)\partial_i\partial_j + \partial_i(h(x)P^{ij}(x))\partial_j\right)\bar{K}(x, 0; \beta) = 0, \quad (2.22)$$

where the projector  $P^{ij}(x)$  and the function  $h(x)$  are given by

$$P^{ij}(x) = \delta_{ij} - \frac{x_i x_j}{x^2}, \quad (2.23)$$

$$h(x) = -\frac{f(x)}{1+f(x)} = \frac{2(Mx)^2}{1-\cos(2Mx)} - 1, \quad (2.24)$$

as discussed in Appendix A. The function  $h(x)$  is a function of only  $x^2 = \delta_{ij}x^i x^j$ , since it is even in  $x \equiv \sqrt{\delta_{ij}x^i x^j}$ . This is a consequence of the maximal symmetry of the sphere. The explicit evaluation of the derivatives appearing in (2.22) produces (recalling the orthogonality condition  $P^{ij}x_j = 0$ )

$$\begin{aligned} h(x)P^{ij}(x)\partial_i\partial_j\bar{K}(x, 0; \beta) &= 2h(x)\delta_{ij}P^{ij}(x)\frac{\partial}{\partial x^2}\bar{K}(x, 0; \beta) \\ &= 2(d-1)h(x)\frac{\partial}{\partial x^2}\bar{K}(x, 0; \beta) \end{aligned} \quad (2.25)$$

and

$$\partial_i(h(x)P^{ij}(x))\partial_j\bar{K}(x, 0; \beta) = -2(d-1)h(x)\frac{\partial}{\partial x^2}\bar{K}(x, 0; \beta). \quad (2.26)$$

The two terms cancel each other, so that we have indeed verified Eq. (2.22) and the correctness of the heat kernel equation (2.19) for our problem.

To summarize, we are led to consider the euclidean Schrödinger equation

$$-\frac{\partial}{\partial\beta}\bar{K}(x, x'; \beta) = \left(-\frac{1}{2}\delta^{ij}\partial_i\partial_j + V_{\text{eff}}(x)\right)\bar{K}(x, x'; \beta), \quad (2.27)$$

valid in Riemann normal coordinates centered at  $x'$ , to describe the quantum motion of a particle on a sphere. This equation can now be solved by a standard path integral

$$\bar{K}(x, x'; \beta) = \int_{x(0)=x'}^{x(\beta)=x} Dx e^{-S[x]}, \quad (2.28)$$

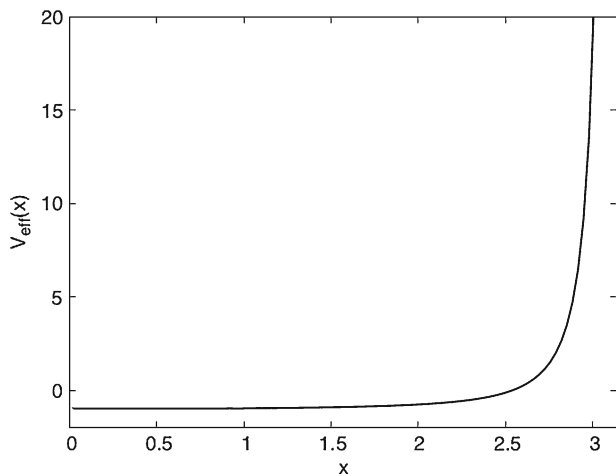
where the action is that of a linear sigma model augmented by an effective potential

$$S[x] = \int_0^\beta dt \left( \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + V_{\text{eff}}(x) \right). \quad (2.29)$$

The required effective potential is

$$V_{\text{eff}}(x) = -\frac{1}{2}g^{-\frac{1}{4}}\partial_i\sqrt{g}g^{ij}\partial_jg^{-\frac{1}{4}} \quad (2.30)$$

and can be computed in terms of the function  $f(x)$  given in (A.4) and (A.8) as



**Fig. 1** Graphical representation of the effective potential for  $d = 4$  and  $M = 1$  (sphere of unit radius)

$$V_{\text{eff}}(x) = \frac{(d-1)}{8} \left[ \frac{(d-5)}{4} \left( \frac{f'(x)}{1+f(x)} \right)^2 + \frac{1}{1+f(x)} \left( \frac{(d-1)}{x} f'(x) + f''(x) \right) \right] \quad (2.31)$$

or, more explicitly, as

$$V_{\text{eff}}(x) = \frac{d(1-d)}{12} M^2 + \frac{(d-1)(d-3)}{48} \times \frac{(5(Mx)^2 - 3 + ((Mx)^2 + 3) \cos(2Mx))}{x^2 \sin^2(Mx)}. \quad (2.32)$$

The point  $x = 0$  is the origin of the Riemann normal coordinates (say the north pole), and we see that the effective potential becomes singular when  $x = \pi/M$ , i.e. at the south pole. We plot the radial behavior of this potential in Fig. 1. Note that the potential is basically flat around  $x = 0$ , but diverges at  $x = \pi/M$  that corresponds to the south pole of the sphere, and which is a coordinate singularity of the patch considered (the so-called normal neighborhood).

### 3 Perturbative expansion

The path integral expression for the transition amplitude on a sphere in terms of the linear sigma model (2.29) is much simpler than the corresponding one with the nonlinear sigma model with lagrangian (2.1). In particular, its perturbative evaluation is straightforward as no perturbative vertices are produced from the kinetic term and from the path integral measure, which is translational invariant. The only vertices are those without derivatives arising from the expansion of  $V_{\text{eff}}$ . They produce Feynman graphs that do not need any regularization.

Let us now describe the perturbative expansion of the transition amplitude (2.28) by considering coinciding initial and final points,  $x' = x$ , which give the diagonal part of the heat kernel  $\bar{K}(x, x; \beta)$ . This is enough to identify one-loop effective actions and anomalies in QFT using worldlines. We must use Riemann normal coordinates centered at  $x' = x$ , and in such coordinates the diagonal heat kernel evaluated at the origin is denoted by  $\bar{K}(0, 0; \beta)$ .

To start with we rescale the time to  $\tau = \frac{t}{\beta}$  to write the action in the form

$$S[x] = \int_0^1 d\tau \left( \frac{1}{2\beta} \delta_{ij} \dot{x}^i \dot{x}^j + \beta V_{\text{eff}}(x) \right), \quad (3.1)$$

which shows that, in an expansion for short times  $\beta$ , the leading behavior is due to the kinetic term, while the effective potential  $V_{\text{eff}}$  gives perturbative corrections. The perturbative expansion of the path integral is obtained by setting

$$S[x] = S_{\text{free}}[x] + S_{\text{int}}[x], \quad (3.2)$$

with

$$S_{\text{free}}[x] = \frac{1}{\beta} \int_0^1 d\tau \frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j, \quad (3.3)$$

$$S_{\text{int}}[x] = \beta \int_0^1 d\tau V_{\text{eff}}(x), \quad (3.4)$$

so that the transition amplitude at coinciding points  $x = x' = 0$  in Riemann normal coordinates may be written as

$$\bar{K}(0, 0; \beta) = \frac{\langle e^{-S_{\text{int}}} \rangle}{(2\pi\beta)^{\frac{d}{2}}}, \quad (3.5)$$

where  $\langle \dots \rangle$  denotes a normalized correlation function with the free path integral weight. The normalization is just the one of a free particle and corresponds to the exact path integral performed with  $S_{\text{free}}$ .

The free propagator for the dynamical variables  $x^i(\tau)$ , vanishing both at  $\tau = 0$  and  $\tau = 1$  (Dirichlet boundary conditions with initial and final point fixed at the origin of the Riemann coordinates), is obtained by inverting the differential operator of the kinetic term in (3.3) and reads

$$\langle x^i(\tau) x^j(\sigma) \rangle = -\beta \delta^{ij} [\partial_\tau^2]_{(\tau, \sigma)}^{-1} = -\beta \delta^{ij} \Delta(\tau, \sigma), \quad (3.6)$$

where the Green function  $\Delta(\tau, \sigma)$  with vanishing Dirichlet boundary conditions is given by

$$\begin{aligned} \Delta(\tau, \sigma) &= (\tau - 1)\sigma \theta(\tau - \sigma) + (\sigma - 1)\tau \theta(\sigma - \tau) \\ &= \frac{1}{2} |\tau - \sigma| - \frac{1}{2}(\tau + \sigma) + \tau\sigma, \end{aligned} \quad (3.7)$$

where  $\theta(x)$  is the Heaviside step function (the regulated value  $\theta(0) = \frac{1}{2}$  is not needed in the evaluation of the perturbative corrections). The two expressions are equivalent, and one

may use the preferred one. The Green function  $\Delta(\tau, \sigma)$  satisfies the defining equation

$$\partial_\tau^2 \Delta(\tau, \sigma) = \delta(\tau - \sigma), \quad (3.8)$$

and the boundary conditions

$$\Delta(0, \sigma) = \Delta(\tau, 0) = 0. \quad (3.9)$$

We are now ready to evaluate perturbative corrections. They are obtained by expanding the effective potential  $V_{\text{eff}}$ , and computing the perturbative terms with an application of the Wick theorem (i.e. calculating gaussian averages). Taylor expanding the potential (2.32) about  $x = 0$  produces

$$\begin{aligned} V_{\text{eff}}(x) = & M^2 \frac{d(1-d)}{12} + M^2(d-1)(d-3) \\ & \times \left( \frac{(Mx)^2}{120} + \frac{(Mx)^4}{756} + \frac{(Mx)^6}{5400} + \right. \\ & + \frac{(Mx)^8}{41580} + \frac{691(Mx)^{10}}{232186500} + \frac{(Mx)^{12}}{2806650} \\ & \left. + \frac{3617(Mx)^{14}}{86837751000} + O(x^{16}) \right), \end{aligned} \quad (3.10)$$

and the interaction vertices arising from it may be written as

$$S_{\text{int}} = \beta \int_0^1 d\tau V_{\text{eff}}(x) = \sum_{m=0}^{\infty} S_{2m}, \quad (3.11)$$

where  $S_{2m}$  is the term containing the power  $(x^2)^m$ , with  $x^2 = \mathbf{x}^2 = x^i x_i$ . Their structure is of the form

$$S_{2m} = \beta M^{2+2m} k_{2m} \int_0^1 d\tau (x^2)^m, \quad (3.12)$$

where the overall power of  $M^2$  has been factored out, while the remaining numerical coefficients  $k_{2m}$  can be read off from (3.10).

The first term

$$S_0 = \beta M^2 \frac{d(1-d)}{12} \quad (3.13)$$

is just a constant, and it can be immediately extracted out of (3.5) to give

$$\overline{K}(0, 0; \beta) = \frac{e^{-S_0 + \dots}}{(2\pi\beta)^{\frac{d}{2}}} \quad (3.14)$$

also written more explicitly in terms of the scalar curvature  $R$  as

$$\begin{aligned} \overline{K}(0, 0; \beta) &= \frac{1}{(2\pi\beta)^{\frac{d}{2}}} \exp \left[ \frac{\beta M^2}{12} d(d-1) + \dots \right] \\ &= \frac{1}{(2\pi\beta)^{\frac{d}{2}}} \exp \left[ \frac{\beta R}{12} + \dots \right]. \end{aligned} \quad (3.15)$$

Notice that this result is exact for the three-sphere  $S^3$ , as for  $d = 3$  the remaining part of the effective potential vanishes. This answer was obtained long ago by Schulman [29], who used the fact that  $S^3$  coincides with the group manifold of  $SU(2)$ .

The next correction is the first nontrivial one, and arises from the vertex  $S_2$ . Expanding the interaction term in (3.5) as

$$\langle e^{-S_{\text{int}}} \rangle = e^{-S_0} (1 - \langle S_2 \rangle + \dots) \quad (3.16)$$

shows that one must compute the correlation function  $\langle S_2 \rangle$ . It identifies a loop graph of the form



where the coefficient of the quadratic vertex is found from (3.10), and the propagator is the one given in (3.7). Its calculation proceeds as follows:

$$\begin{aligned} -\langle S_2 \rangle &= -\beta M^4 k_2 \int_0^1 d\tau \delta_{ij} \langle x^i(\tau) x^j(\tau) \rangle \\ &= \beta^2 M^4 k_2 d \int_0^1 d\tau \Delta(\tau, \tau) \\ &= -\frac{\beta^2 M^4}{720} d(d-1)(d-3), \end{aligned} \quad (3.17)$$

as the coupling  $k_2$  obtained from (3.10) is  $k_2 = \frac{(d-1)(d-3)}{120}$ , while the integral of the two-point correlation function at coinciding points ( $\sigma = \tau$ ) gives

$$\int_0^1 d\tau \Delta(\tau, \tau) = \int_0^1 d\tau (\tau^2 - \tau) = -\frac{1}{6}. \quad (3.18)$$

This result can be exponentiated to account for the disconnected contributions arising from such graphs at higher orders. Thus, at this perturbative level, we find the transition amplitude

$$\overline{K}(0, 0; \beta) = \frac{e^{-S_0}}{(2\pi\beta)^{\frac{d}{2}}} e^{-\langle S_2 \rangle + O(\beta^3)}, \quad (3.19)$$

which takes the explicit form

$$\begin{aligned}\bar{K}(0, 0; \beta) &= \frac{1}{(2\pi\beta)^{\frac{d}{2}}} \exp \left[ \frac{\beta M^2}{12} d(d-1) \right. \\ &\quad \left. - \frac{\beta^2 M^4}{720} d(d-1)(d-3) + \dots \right] \\ &= \frac{1}{(2\pi\beta)^{\frac{d}{2}}} \exp \left[ \frac{\beta R}{12} - \frac{(\beta R)^2}{6!} \frac{(d-3)}{d(d-1)} + \dots \right].\end{aligned}\quad (3.20)$$

In a similar way one may proceed to higher orders. It is clear that all perturbative corrections appear as powers of  $\beta M^2$ , or equivalently  $\beta R$ , as verified by power counting. In this section we wish to reach order  $\beta^8$ , so that we must compute

$$\begin{aligned}\bar{K}(0, 0; \beta) &= \frac{e^{-S_0}}{(2\pi\beta)^{\frac{d}{2}}} \exp \left[ -\langle S_2 \rangle - \underbrace{\langle S_4 \rangle}_{O(\beta^3)} \right. \\ &\quad \underbrace{-\langle S_6 \rangle + \frac{1}{2} \langle S_2^2 \rangle_c}_{O(\beta^4)} \\ &\quad \underbrace{-\langle S_8 \rangle + \langle S_4 S_2 \rangle_c}_{O(\beta^5)} \\ &\quad \underbrace{-\langle S_{10} \rangle + \langle S_6 S_2 \rangle_c + \frac{1}{2} \langle S_4^2 \rangle_c - \frac{1}{3!} \langle S_2^3 \rangle_c}_{O(\beta^6)} \\ &\quad \underbrace{-\langle S_{12} \rangle + \langle S_8 S_2 \rangle_c + \langle S_6 S_4 \rangle_c - \frac{1}{2} \langle S_4 S_2^2 \rangle_c}_{O(\beta^7)} \\ &\quad \underbrace{-\langle S_{14} \rangle + \langle S_{10} S_2 \rangle_c + \langle S_8 S_4 \rangle_c + \frac{1}{2} \langle S_6^2 \rangle_c - \frac{1}{2} \langle S_4^2 S_2 \rangle_c - \frac{1}{2} \langle S_6 S_2^2 \rangle_c + \frac{1}{4!} \langle S_2^4 \rangle_c}_{O(\beta^8)} \\ &\quad \left. + O(\beta^9) \right].\end{aligned}\quad (3.21)$$

The calculation up to  $O(\beta^6)$  was sketched in [1]. Here we continue through order  $O(\beta^7)$  and  $O(\beta^8)$ . As indicated by the notation  $\langle \dots \rangle_c$ , it is enough to compute connected correlation functions only, as the disconnected pieces have been automatically included by exponentiation. We report the detailed calculations in Appendix B.

Adding all contributions, we summarize our final result for the heat kernel at coinciding points

$$\begin{aligned}\bar{K}(0, 0; \beta) &= \frac{1}{(2\pi\beta)^{\frac{d}{2}}} \exp \left[ d(d-1) \frac{\beta M^2}{12} \right. \\ &\quad + d(d-1)(d-3) \left( -\frac{(\beta M^2)^2}{720} \right. \\ &\quad \left. - \frac{(\beta M^2)^3}{7!} \frac{2(d+2)}{9} \right. \\ &\quad \left. - \frac{(\beta M^2)^4}{7!} \frac{(d^2+20d+15)}{360} \right. \\ &\quad \left. + \frac{(\beta M^2)^5}{11!} \frac{8(d+2)(d^2-12d-9)}{3} \right. \\ &\quad \left. + \frac{(\beta M^2)^6}{13!} \frac{8(1623d^4-716d^3-65930d^2-123572d-60165)}{315} \right. \\ &\quad \left. + \dots \right]\end{aligned}$$

$$\begin{aligned}&+ \frac{(\beta M^2)^7}{13!} \frac{16(d+2)(33d^4+404d^3-2510d^2-6612d-3915)}{315} \\ &- \frac{(\beta M^2)^8}{17!} \frac{8}{45} (12405d^6-810668d^5-1953995d^4 \\ &+ 17853784d^3+71217159d^2 \\ &+ 92279700d+40157775) + O(\beta^9) \Big],\end{aligned}\quad (3.22)$$

which we present also in terms of the scalar curvature  $R$  (recall that  $R = M^2 d(d-1)$  with  $M = \frac{1}{a}$  the inverse sphere radius)

$$\begin{aligned}\bar{K}(0, 0; \beta) &= \frac{1}{(2\pi\beta)^{\frac{d}{2}}} \exp \left[ \frac{\beta R}{12} - \frac{(\beta R)^2}{6!} \frac{(d-3)}{d(d-1)} \right. \\ &\quad - \frac{(\beta R)^3}{9!} \frac{16(d-3)(d+2)}{d^2(d-1)^2} \\ &\quad - \frac{(\beta R)^4}{10!} \frac{2(d-3)(d^2+20d+15)}{d^3(d-1)^3} \\ &\quad + \frac{(\beta R)^5}{11!} \frac{8(d-3)(d+2)(d^2-12d-9)}{3d^4(d-1)^4} \\ &\quad + \frac{(\beta R)^6}{13!} \frac{8(d-3)(1623d^4-716d^3-65930d^2-123572d-60165)}{315d^5(d-1)^5} \\ &\quad + \frac{(\beta R)^7}{14!} \frac{32(d-3)(d+2)(33d^4+404d^3-2510d^2-6612d-3915)}{45d^6(d-1)^6} \\ &\quad - \frac{(\beta R)^8}{17!} \frac{8(d-3)}{45d^7(d-1)^7} (12405d^6-810668d^5 \\ &\quad - 1953995d^4+17853784d^3 \\ &\quad + 71217159d^2+92279700d+40157775) + O(\beta^9) \Big].\end{aligned}\quad (3.23)$$

This exponential can be expanded keeping terms up to order  $O(\beta^8)$  included, to read off the heat kernel coefficients at coinciding points  $a_n(0, 0)$  for the integer  $n$  up to  $n = 8$ , defined by

$$\bar{K}(0, 0; \beta) = \frac{1}{(2\pi\beta)^{\frac{d}{2}}} \sum_{n=0}^{\infty} a_n(0, 0) \beta^n. \quad (3.24)$$

We will do this in the next section for conformal hamiltonians to extract the so-called type-A trace anomalies.

#### 4 The type-A trace anomalies

One may use the path integral calculation of the transition amplitude on a sphere to evaluate the type-A trace anomalies of a conformal scalar field. We have performed this exercise in [1] to test the correctness and usefulness of the linear sigma model approach. We are now ready to extend those results to identify the trace anomalies in  $d = 14$  and  $d = 16$  dimensions. As reviewed in [1], the trace anomaly of the conformal scalar field can be related to the transition amplitude of a particle in a curved space by

$$\langle T^\mu{}_\mu(x) \rangle_{\text{QFT}} = \lim_{\beta \rightarrow 0} K_\xi(x, x; \beta), \quad (4.1)$$



where on the left hand side  $T^\mu{}_\mu(x)$  is the trace of the stress tensor of the conformal scalar in a curved background, and the expectation value is performed in the corresponding quantum field theory. The right hand side can instead be viewed as the anomalous contribution arising from the QFT path integral measure regulated *à la* Fujikawa [30]. The regulator  $H_\xi$  corresponds to the kinetic term of the scalar quantum field theory, which is proportional to the conformal laplacian. This is identified with the quantum hamiltonian  $H_\xi$  of a particle in a curved space

$$H_\xi = \frac{1}{2}(-\nabla^2 + \xi R), \quad \xi = \frac{(d-2)}{4(d-1)}, \quad (4.2)$$

which must be used in evaluating the transition element  $K_\xi(x, x; \beta)$  at coinciding points [9, 10]. It is understood that the  $\beta \rightarrow 0$  limit in (4.1) picks up just the  $\beta$ -independent term, as divergent terms are removed by the QFT renormalization. This procedure selects the appropriate heat kernel coefficient  $a_n(x, x)$  sitting in the expansion of  $K_\xi(x, x; \beta)$ , as in (3.24). To reproduce the correct conformal hamiltonian (4.2) we must add a nonminimal coupling through an additional constant potential

$$V_\xi = \frac{1}{2}\xi R, \quad (4.3)$$

so that we must shift  $V_{\text{eff}} \rightarrow V_{\text{eff}} + V_\xi$  in (3.4). Its effect is to replace the leading term of (3.23) by

$$\frac{1}{(2\pi\beta)^{\frac{d}{2}}} \exp\left[\frac{\beta R}{12} + \dots\right] \rightarrow \frac{1}{(2\pi\beta)^{\frac{d}{2}}} \exp\left[\frac{\beta}{12}(1-6\xi)R + \dots\right] \quad (4.4)$$

to obtain the desired amplitude  $K_\xi(x, x; \beta)$ . Expanding  $K_\xi(x, x; \beta)$  at the required order we find the trace anomalies in  $d$  dimensions

$$\langle T^\mu{}_\mu(x) \rangle_{\text{QFT}} = \frac{a_{\frac{d}{2}}(x, x)}{(2\pi)^{\frac{d}{2}}}, \quad (4.5)$$

which we list in Table 1, expressing the results also in terms of the sphere radius  $a = \frac{1}{M}$ . Of course, one may use Riemann normal coordinates centered at  $x$ , so that  $\sqrt{g(x)} = 1$  and the result in (3.23) is directly applicable.

## 5 Alternative methods and checks

In the present section we check our results on the type-A trace anomalies of a conformal scalar by using alternative approaches based on the  $\zeta$ -function regularization. One

**Table 1** Type-A trace anomalies of a conformal scalar field

$d$	$\langle T^\mu{}_\mu \rangle$	$\langle T^\mu{}_\mu \rangle$
2	$\frac{R}{24\pi}$	$\frac{1}{12\pi a^2}$
4	$-\frac{R^2}{34560\pi^2}$	$-\frac{1}{240\pi^2 a^4}$
6	$\frac{R^3}{21772800\pi^3}$	$\frac{5}{4032\pi^3 a^6}$
8	$-\frac{23R^4}{339880181760\pi^4}$	$-\frac{23}{34560\pi^4 a^8}$
10	$\frac{263R^5}{2993075712000000\pi^5}$	$\frac{263}{506880\pi^5 a^{10}}$
12	$-\frac{133787R^6}{1330910037208675123200\pi^6}$	$-\frac{133787}{251596800\pi^6 a^{12}}$
14	$\frac{157009R^7}{1536182179466286307737600\pi^7}$	$\frac{157009}{232243200\pi^7 a^{14}}$
16	$-\frac{16215071R^8}{1738368537956293017600000000\pi^8}$	$-\frac{16215071}{15792537600\pi^8 a^{16}}$

method was used in [14] and later re-elaborated in [15]. Following the prescription reported in those references, one finds that the type-A trace anomaly on a  $d$ -sphere is given by

$$\langle T^\mu{}_\mu \rangle = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{\frac{d+1}{2}}a^d} \zeta_{Y_d}(0), \quad (5.1)$$

where  $\zeta_{Y_d}(s)$  is the  $\zeta$ -function associated to the (eigenvalues of the) kinetic operator  $Y_d$  of the conformally coupled scalar field on the sphere

$$Y_d = -\nabla^2 + \frac{(d-2)}{4(d-1)}R, \quad (5.2)$$

often called “Yamabe operator” in the mathematical literature. Its analytic continuation at  $s \rightarrow 0$  is given by<sup>1</sup>

$$\zeta_{Y_d}(0) = \frac{1}{(d-1)!} \sum_{p=0}^{(d-2)/2} \frac{C_p(d)}{p+1} \left\{ \left(1 - 2^{-(2p+1)}\right) B_{2p+2} - 2^{-2p} \left(\frac{1}{2}p + \frac{1}{4}\right) \right\}, \quad (5.3)$$

where  $B_{2p+2}$  are Bernoulli numbers. The set of numerical coefficients  $C_p(d)$  is the solution of the linear system

$$\frac{(n+d-2)!}{n!} = \sum_{p=0}^{(d-2)/2} C_p(d) \left(n + \frac{d-1}{2}\right)^{2p}, \quad n = 1, \dots, \frac{d}{2}. \quad (5.4)$$

For  $d = 14, 16$  they read

<sup>1</sup> The expression below coincides with Eq. (2.29) of [14], thanks to the identity  $\sum_{p=0}^{(d-2)/2} C_p(d) 2^{-2p} = 0$ , which is satisfied by the values of  $C_p(d)$  of Table 2, as one may check.

**Table 2** Solutions of the linear system (5.4), for  $d = 14, 16$ 

$C_p(d)$	0	1	2	3	4	5	6	7
$d = 14$	$\frac{108056025}{4096}$	$-\frac{64408383}{512}$	$\frac{21967231}{256}$	$-\frac{308737}{16}$	$\frac{28743}{16}$	$-\frac{143}{2}$	1	*
$d = 16$	$-\frac{18261468225}{16384}$	$\frac{21878089479}{4096}$	$-\frac{3841278805}{1024}$	$\frac{230673443}{256}$	$-\frac{6092515}{64}$	$\frac{77077}{16}$	$-\frac{455}{4}$	1

which, inserted into (5.3), yield

$$\zeta_{Y_d}(0) = \begin{array}{c|cc} d & 14 & 16 \\ \hline \zeta_{Y_d}(0) & \frac{157009}{122594472000} & -\frac{16215071}{62523180720000} \end{array} \quad (5.5)$$

and using these values into the general expression (5.1) produces the type-A trace anomalies that match our results of Table 1.

More recently, within the AdS/CFT paradigm, it was shown how to directly reproduce the  $\zeta$ -function for a class of conformal operators [16]—see also [17] for a direct proof that does not use holography. For the quadratic operator  $Y_d$  this amounts to compute the following integral:

$$\zeta_{Y_d}(0) = \frac{2(-)^{d/2}}{d!} \int_0^1 dv \prod_{l=0}^{d/2-1} (l^2 - v^2), \quad d > 2, \quad (5.6)$$

which can easily be checked to reproduce (5.5).

## 6 Conclusion and outlook

Mastering the computation of scattering amplitudes that involve gravitons is an outstanding task that keeps drawing the attention of many theoretical physicists—recently, for example, several interesting papers have dealt with the issue of soft graviton insertions in scattering amplitudes; see for example [31–35] or the pedagogical review [36]. From the worldline formalism viewpoint, the main difficulty in tackling the computation of graviton scattering amplitudes resides in the presence of derivative interactions in the nonlinear sigma model, which represents the first quantized particle in a generically curved space. In the present paper, following the developments of Ref. [1], we have investigated further the use of an effective linear sigma model to study the one-loop effective action of a scalar field in a maximally symmetric curved space, and its type-A trace anomaly in particular. In the literature, other and certainly more efficient methods to compute type-A trace anomalies of conformal QFTs are known—often based on the  $\zeta$ -function approach to computing determinants. However, unlike those methods, the present approach is much more flexible, allowing for example to compute the off-diagonal parts of the heat kernel and, in general, to give a worldline representation of the QFT observable that one wishes to study, see for example the recent use of

a worldline representation to relate different quantities made in [37].

To extend further the use of the linear sigma model approach, it would be interesting to prove its validity on arbitrary geometries, a possibility already envisaged in [12], but whose implementation might be obstructed by backgrounds with less symmetries than the maximal one.

Considering only spaces with maximal symmetries, a still useful extension would be the introduction of worldline fermions, so to be able to consider  $N = 1$  and  $N = 2$  supersymmetric generalizations, as needed in the worldline description of spin 1/2 and spin 1 particles. An extension to arbitrary  $N$  would also allow one to study higher spinning particles on maximally symmetric spaces [38, 39]. In the nonlinear sigma model approach the regularizations and counterterms for the supersymmetric version of the path integral of a particle in a curved space have been most extensively analyzed at arbitrary  $N$  in [40]. A linear sigma model approach would carry many simplifications and would certainly be welcome. In the case of spin 1/2, one might wish to study from a worldline perspective the issue of the trace anomaly of a Weyl fermion, where an apparent clash between the results of [41, 42] and [43] has emerged. However, to address that point with worldline methods requires mastering the use of a generic background, as the conflicting result sits in the coefficient of a type-B trace anomaly.

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## Appendix A: Geometry of maximally symmetric spaces and Riemann normal coordinates

Maximally symmetric spaces are those that have a maximal number of isometries, namely  $d(d+1)/2$  for a  $d$ -dimensional space. Their curvature tensors can be expressed in terms of the metric as

$$R_{ijmn} = M^2(g_{im}g_{jn} - g_{in}g_{jm}), \quad (A.1)$$

$$R_{ij} = R_{mi}{}^m{}_j = M^2(d-1)g_{ij}, \quad (A.2)$$

$$R = R_i{}^i = M^2(d-1)d, \quad (A.3)$$



where  $M^2$  is a constant which identifies the sectional curvature of the manifold. This constant is positive on a sphere of radius  $a$ , where  $M^2 = 1/a^2$ , it vanishes for a flat space, and it is negative for a real hyperbolic space. This exhausts the list of maximally symmetric spaces. For simplicity in the main text we have considered spheres, but here we treat briefly real hyperbolic spaces as well.

In the main text we use Riemann normal coordinates (for details see [44, 45], and [46–48] for their application to non-linear sigma models; the most accurate and explicit expansion of the metric around the origin that we are aware of may be found in [49]). On spheres the sectional curvature is positive, and we can take  $M = \frac{1}{a} > 0$ . It is then easy to evaluate recursively all terms in the expansion of the metric [50]

$$g_{ij}(x) = \delta_{ij} + \sum_{l=1}^{\infty} c_l M^{2l} (-1)^l (x^2)^l P_{ij} = \delta_{ij} + f(x) P_{ij}, \quad (\text{A.4})$$

where the  $x^i$  denote now the Riemann normal coordinates centered around a point (the origin),  $P_{ij}$  indicates a projector given by

$$P_{ij} = \delta_{ij} - \hat{x}_i \hat{x}_j, \quad \hat{x}^i = \frac{x^i}{x}, \quad x = \sqrt{\mathbf{x}^2}, \quad (\text{A.5})$$

and  $c_l$  are coefficients that obey the recursion relation

$$c_l = \frac{2}{(l+1)(2l+1)} c_{l-1}, \quad c_0 = 1, \quad (\text{A.6})$$

and they are found to be given by

$$c_l = 2 \frac{4^l}{(2l+2)!}. \quad (\text{A.7})$$

The series can be summed up to give

$$f(x) = \frac{1 - 2(Mx)^2 - \cos(2Mx)}{2(Mx)^2}, \quad (\text{A.8})$$

which was also reproduced in [50] (there is a misprint in Eq. (11) of [50], where a factor  $(x^2)^2$  in the denominator should be replaced by  $x^2$ ). The present manuscript has the correct answer. Note that the function  $f(x)$  does not have poles and it is even in  $x$ , so that it depends only on  $x^2 = \mathbf{x}^2 = \delta_{ij} x^i x^j$ . Note also that because of the projector  $P_{ij}$  one has the equality  $x^2 = g_{ij}(x) x^i x^j$ . It is now immediate to compute the inverse metric  $g^{ij}(x)$  and the metric determinant  $g(x)$  as

$$g^{ij}(x) = \delta^{ij} + h(x) P^{ij}, \quad (\text{A.9})$$

$$g(x) = (1 + f(x))^{d-1}, \quad (\text{A.10})$$

where

$$h(x) = -\frac{f(x)}{1 + f(x)}. \quad (\text{A.11})$$

We recall again that on the right hand side of these formulas indices are raised and lowered with the flat metric  $\delta_{ij}$ .

For completeness, we discuss the case of real hyperbolic spaces as well. Now the sectional curvature is negative,  $M^2 < 0$ . It can be obtained from the previous case by the analytic continuation  $M \rightarrow i|M|$ , with the imaginary unit  $i$  giving rise to the negative sign of the sectional curvature, and  $|M| = \sqrt{-M^2}$ . Performing this analytic continuation in (A.4) we find that in the sum the minus signs from  $(-1)^l$  get canceled

$$g_{ij}(x) = \delta_{ij} + \sum_{l=1}^{\infty} c_l |M|^{2l} (x^2)^l P_{ij} = \delta_{ij} + f(x) P_{ij}, \quad (\text{A.12})$$

and the sum now converges to the function

$$f(x) = \frac{-1 - 2(|M|x)^2 + \cosh(2|M|x)}{2(|M|x)^2}. \quad (\text{A.13})$$

Finally, the function  $f(x)$  vanishes in the flat space case, where Riemann normal coordinates are just the standard cartesian coordinates. It may also be obtained as a smooth limit of the curved cases, as  $f(x) \rightarrow 0$  for  $M \rightarrow 0$ .

## Appendix B: Perturbative calculations

We describe here the perturbative calculations needed to identify the corrections of order  $\beta^7$  and  $\beta^8$  to the transition amplitude (3.21). Lower orders have been computed in [1].

At order  $\beta^7$  we need to evaluate

$$-\langle S_{12} \rangle + \langle S_8 S_2 \rangle_c + \langle S_6 S_4 \rangle_c - \frac{1}{2} \langle S_2^2 S_4 \rangle_c \quad (\text{B.1})$$

where  $\langle \dots \rangle_c$  indicates connected correlation functions. In reporting our intermediate results we set  $M = 1$  (sphere of unit radius), use the abbreviation  $\Delta(\tau_1, \tau_2) \equiv \Delta_{12}$  for the propagator, and indicate the contributions from topologically distinct Wick contractions that give rise to different powers of the dimension  $d$ , to assist in the verification of our intermediate results.

The different contributions are as follows:

$$-\langle S_{12} \rangle = -\beta^7 k_{12} \underbrace{(d^6 + 30d^5 + 340d^4 + 1800d^3 + 4384d^2 + 3840d)}_{d(d+2)(d+4)(d+6)(d+8)(d+10)} \underbrace{\int_0^1 d\tau_1 \Delta_{11}^6}_{\frac{1}{12012}} \quad (\text{B.2})$$

$$\langle S_8 S_2 \rangle_c = -\beta^7 k_8 k_2 \underbrace{\left( 8d^4 + 96d^3 + 352d^2 + 384d \right)}_{8d(d+2)(d+4)(d+6)} \underbrace{\int_0^1 d\tau_1 \int_0^1 d\tau_2 \Delta_{12}^2 \Delta_{11}^3}_{-\frac{1}{8316}}, \quad (\text{B.3})$$

$$\langle S_6 S_4 \rangle_c = -\beta^7 k_6 k_4 \left( \underbrace{\left( 12d^4 + 96d^3 + 240d^2 + 192d \right)}_{12d(d+2)^2(d+4)} \underbrace{\int_0^1 d\tau_1 \int_0^1 d\tau_2 \Delta_{11}^2 \Delta_{12}^2 \Delta_{22}}_{-\frac{2}{17325}} + \underbrace{\left( 24d^3 + 144d^2 + 192d \right)}_{24d(d+2)(d+4)} \underbrace{\int_0^1 d\tau_1 \int_0^1 d\tau_2 \Delta_{11} \Delta_{12}^4}_{-\frac{1}{13860}} \right) \quad (\text{B.4})$$

$$-\frac{1}{2} \langle S_2^2 S_4 \rangle_c = -\beta^7 k_2^2 k_4 4d(d+2) \left( 2 \underbrace{\iiint \Delta_{12} \Delta_{13} \Delta_{23} \Delta_{33}}_{\frac{13}{56700}} + \underbrace{\iiint \Delta_{13}^2 \Delta_{23}^2}_{\frac{1}{5670}} \right). \quad (\text{B.5})$$

At order  $\beta^8$  we need instead

$$-\langle S_{14} \rangle + \langle S_{10} S_2 \rangle_c + \langle S_8 S_4 \rangle_c + \frac{1}{2} \langle S_6^2 \rangle_c - \frac{1}{2} \langle S_4^2 S_2 \rangle_c - \frac{1}{2} \langle S_6 S_2^2 \rangle_c + \frac{1}{4!} \langle S_2^4 \rangle_c, \quad (\text{B.6})$$

and the different contributions are now as follows:

$$-\langle S_{14} \rangle = \beta^8 k_{14} \underbrace{\left( d^7 + 42d^6 + 700d^5 + 5880d^4 + 25984d^3 + 56448d^2 + 46080d \right)}_{d(d+2)(d+4)(d+6)(d+8)(d+10)(d+12)} \underbrace{\int_0^1 d\tau_1 \Delta_{11}^7}_{-\frac{1}{51480}}, \quad (\text{B.7})$$

$$\langle S_{10} S_2 \rangle_c = \beta^8 k_{10} k_2 \underbrace{\left( 10d^5 + 200d^4 + 1400d^3 + 4000d^2 + 3840d \right)}_{10d(d+2)(d+4)(d+6)(d+8)} \underbrace{\int_0^1 d\tau_1 \int_0^1 d\tau_2 \Delta_{11}^4 \Delta_{12}^2}_{\frac{1}{36036}}, \quad (\text{B.8})$$

$$\langle S_8 S_4 \rangle_c = \beta^8 k_8 k_4 \left( \underbrace{\left( 16d^5 + 224d^4 + 1088d^3 + 2176d^2 + 1536d \right)}_{16d(d+2)^2(d+4)(d+6)} \right.$$

$$\underbrace{\int_0^1 d\tau_1 \int_0^1 d\tau_2 \Delta_{11}^3 \Delta_{12}^2 \Delta_{22}}_{\frac{19}{720720}} + \underbrace{\left( 48d^4 + 576d^3 + 2112d^2 + 2304d \right)}_{48d(d+2)(d+4)(d+6)} \underbrace{\int_0^1 d\tau_1 \int_0^1 d\tau_2 \Delta_{11}^2 \Delta_{12}^4}_{\frac{1}{60060}} \right), \quad (\text{B.9})$$

$$\frac{1}{2} \langle S_6^2 \rangle_c = \frac{1}{2} \beta^8 k_6^2 \left( \underbrace{\left( 18d^3(d+2)^2 + 144d^2(d+2)^2 + 288d(d+2)^2 \right)}_{18d(d+2)^2(d+4)^2} \underbrace{\iint \Delta_{12}^2 \Delta_{11}^2 \Delta_{22}^2}_{\frac{491}{18918900}} + \underbrace{\left( 72d^3(d+2) + 576d^2(d+2) + 1152d(d+2) \right)}_{72d(d+2)(d+4)^2} \underbrace{\iint \Delta_{12}^4 \Delta_{11} \Delta_{22}}_{\frac{25}{1513512}} + \underbrace{\left( 48d^3 + 288d^2 + 384d \right)}_{48d(d+2)(d+4)} \underbrace{\iint \Delta_{12}^6}_{\frac{1}{84084}} \right), \quad (\text{B.10})$$

$$-\frac{1}{2} \langle S_4^2 S_2 \rangle_c = \frac{1}{2} \beta^8 k_4^2 k_2 \left( 32d(d+2)^2 \left( \underbrace{\iiint \Delta_{12}^2 \Delta_{23}^2 \Delta_{33}}_{-\frac{2}{51975}} + \underbrace{\iiint \Delta_{12} \Delta_{13} \Delta_{23} \Delta_{22} \Delta_{33}}_{-\frac{83}{1663200}} \right) + 64d(d+2) \underbrace{\iiint \Delta_{12} \Delta_{13} \Delta_{23}^3}_{-\frac{1}{34650}} \right), \quad (\text{B.11})$$

$$-\frac{1}{2}\langle S_6 S_2^2 \rangle_c = \frac{1}{2}\beta^8 k_6 k_2^2 24d(d+2)(d+4) \left( \underbrace{\iiint \Delta_{11}^2 \Delta_{12} \Delta_{13} \Delta_{23}}_{-\frac{8}{155925}} + \underbrace{\iiint \Delta_{11} \Delta_{12}^2 \Delta_{13}^2}_{-\frac{1}{24948}} \right), \quad (\text{B.12})$$

$$\frac{1}{4!}\langle S_2^4 \rangle_c = \frac{1}{4!}\beta^8 k_2^4 48d \underbrace{\iiint \Delta_{12} \Delta_{23} \Delta_{34} \Delta_{41}}_{\frac{1}{9450}}. \quad (\text{B.13})$$

We may now insert the values of the coupling constants

$$\begin{aligned} k_2 &= (d-1)(d-3)\frac{1}{120}, \\ k_4 &= (d-1)(d-3)\frac{1}{756}, \\ k_6 &= (d-1)(d-3)\frac{1}{5400}, \\ k_8 &= (d-1)(d-3)\frac{1}{41580}, \\ k_{10} &= (d-1)(d-3)\frac{691}{232186500}, \\ k_{12} &= (d-1)(d-3)\frac{1}{2806650}, \\ k_{14} &= (d-1)(d-3)\frac{3617}{86837751000}, \end{aligned} \quad (\text{B.14})$$

found from (3.10), reintroduce the correct power of  $M$ , and add all terms to find the final answer reported in (3.22) or equivalently in (3.23).

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